

Kinetic solutions for nonlocal stochastic conservation laws

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Abstract This work is devoted to examine the uniqueness and existence of kinetic solutions for a class of scalar conservation laws involving a nonlocal super-critical diffusion operator and a multiplicative noise. Our proof for uniqueness is based upon the analysis on double variables method and the existence is enabled by a parabolic approximation.

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1 Introduction

The present paper is concerned with the anomalous diffusion related to the Lévy flights [18, 19, 10]. At the macroscopic modeling level, this means the Laplacian for normal diffusion is replaced by a fractional power of the (negative) Laplacian. We consider the following partial differential equation, coupling a conservation law with an anomalous diffusion:

$$\partial_t u(x, t) + \nu(-\Delta_x)^{\frac{\alpha}{2}} u + \operatorname{div}(A(u)) = \Phi(u) \partial_t W(t), \quad x \in \mathbb{R}^d, \quad t \in (0, T), \quad (1.1)$$

with initial data:

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \quad (1.2)$$

where ν is a nonnegative parameter, $\alpha \in (0, 1)$, and $A = (A_1, \dots, A_d)$, a vector field (*the flux*), is supposed to be of class C^2 and its derivatives have at most polynomial growth. Following [6], we assume that W is a cylindrical Wiener process: $W = \sum_{k \geq 1} \beta_k e_k$, where β_k are independent Brownian process and $\{e_k\}_{k \geq 1}$ is a complete orthonormal basis in a Hilbert space H . For each

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$u \in \mathbb{R}$, $\Phi(u) : H \rightarrow L^2(\mathbb{R}^d)$ is defined by $\Phi(u)e_k = g_k(\cdot, u)$, where $g_k(\cdot, u)$ is a regular function on \mathbb{R}^d . More precisely, we assume $g_k \in C(\mathbb{R}^{d+1})$ with the bounds

$$G^2(x, u) = \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0(\hat{g}(x) + |u|^2), \quad (1.3)$$

$$\sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq D_1(|x - y|^2 + |u - v|h(|u - v|)), \quad (1.4)$$

where $0 \leq \hat{g}(x) \in L^1(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$, $u, v \in \mathbb{R}$, and h is a continuous non-decreasing function on \mathbb{R}_+ with $h(0) = 0$.

We briefly mention some recent works on well-posedness of (1.1)-(1.2), which are relevant for the present paper. To add a stochastic forcing $\Phi(u)dw(t)$ is natural for applications, which appears in wide variety of field as physics, engineering, biology and so on. We first recall some results on the stochastic scalar conservation law without diffusion ($\nu = 0$):

$$\partial_t u(x, t) + \operatorname{div}_x A(u) = \Phi(u) \partial_t W(t), \quad x \in \mathbb{R}^d, \quad t \in (0, T). \quad (1.5)$$

The Cauchy problem of equation (1.5) with additive noise has been studied in [13], where J. U. Kim proposed a method of compensated compactness to prove, via vanishing viscosity approximation, the existence of a stochastic weak entropy solution. A Kruzhkov-type method was used to prove the uniqueness. Vallet-Wittbold [17] extended the results of Kim to the multi-dimensional Dirichlet problem with additive noise. By using vanishing viscosity method, Young measure techniques and Kruzhkov doubling variables technique, they proved the existence and uniqueness of the stochastic entropy solution.

Concerning multiplicative noise, for Cauchy problem, Feng-Nualart [11] introduced a notion of strong entropy solution in order to prove the uniqueness for the entropy solution of (1.5). Using the vanishing viscosity and compensated compactness arguments, they established the existence of stochastic strong entropy solution only in 1D case. Chen et al. [5] proved that the multi-dimensional stochastic problem is well-posedness by using a uniform spatial BV-bound. Following the idea of [11, 5], Lv et al. [15] considered the Cauchy problem (1.1). Bauzet et al. [2] proved a result of existence and uniqueness of the weak measure-valued entropy solution to the multi-dimensional Cauchy problem (1.5).

Using a kinetic formulation, Debussche-Vovelle [6] obtained a result of existence and uniqueness of the entropy solution to the problem posed in a d-dimensional torus. About the Cauchy-Dirichlet problem (1.5), see [3].

When $\nu > 0$, the problem (1.1) with (1.2) has been studied in [15, 16], where the Kruzhkov's semi-entropy formulations was used. It is remarked that except for [6, 11], the previous results only considered the Brownian motion perturbation. That is, the noise does not depend on the spatial variable.

Inspired by [6], in this paper, we reconsider the problem (1.1) with (1.2) and obtain the well-posedness by using the kinetic formulation. The advantage of kinetic formulation method is that we can deal with the cylindrical Wiener process in any dimension. We remark that it is not trivial to generalize the results of [6] to the problem (1.1) with (1.2). Because of the nonlocal term $(-\Delta_x)^{\frac{\alpha}{2}} u$, the proof of existence for kinetic solutions will become more complicated and the assumptions on the initial data will become stronger. Moreover, compared with [6], we have to introduce another non-negative measure to overcome the difficulty. The proof of uniqueness of solution to (1.1) will be different from that in (1.5). In this paper, we mainly focus on how to deal with the nonlocal term.

This paper is organized as follows. In Section 2, we introduce some notions on solutions for (1.1)-(1.2), and then prove the uniqueness and existence of kinetic solutions in Section 3. We further discuss the regularity properties and continuous dependence (on nonlinearities and Lévy measures) for kinetic solutions in Section 4.

2 Entropy solutions and kinetic solutions

In this section, we first give the definitions of stochastic entropy solutions and stochastic kinetic solutions, then prove that they are equivalent and last state out our main results.

To present our formulation for (1.1), we recall the following results on the operator $(-\Delta)^{\frac{\alpha}{2}}$.

Lemma 2.1 ([9]) *There exists a constant $C_d(\alpha) > 0$ that only depends on d and α , and such that for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, all $r > 0$ and all $x \in \mathbb{R}^d$*

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}} \phi(x) &= -C_d(\alpha) \int_{|z| \geq r} \frac{\phi(x+z) - \phi(x)}{|z|^{d+\alpha}} dz \\ &\quad - C_d(\alpha) \int_{|z| \leq r} \frac{\phi(x+z) - \phi(x) - \nabla \phi(x) \cdot z}{|z|^{d+\alpha}} dz. \end{aligned}$$

Moreover, when $\alpha \in (0, 1)$, one can take $r = 0$.

We take $\nu = 1$ in Sections 2 and 3. Here and in the followings, we use (\cdot, \cdot) to denote the inner product of L^2 -valued functions. Following [11], we have the definition.

Definition 2.1 (Stochastic Nonlocal Entropy solution) *An $L^2(\mathbb{R}^d)$ -valued $\{\mathcal{F}_t : 0 \leq t \leq T\}$ -predictable stochastic process $\{u(t) = u(x, t)\}$ is called a stochastic entropy solution of (1.1) provided*

(1) *For each $p \geq 1$*

$$\mathbb{E}[\text{ess sup}_{0 \leq t \leq T} \|u(t)\|_p^p] < \infty;$$

(2) *For $0 \leq \psi \in C_c^{1,2}([0, T] \times \mathbb{R}^d)$ and all convex $\eta \in C^2(\mathbb{R})$, the following inequality holds*

$$\begin{aligned} &\int_0^T (\eta(u(r)), \partial_t \psi(r, \cdot)) dr + (\eta(u(0)), \psi(0, \cdot)) + \int_0^T (\Psi(u(r)), \nabla_x \psi(r, \cdot)) dr \\ &- \int_0^T (\eta(u(r)), (-\Delta)^{\frac{\alpha}{2}} \psi(r)) dr + \sum_{k \geq 1} \int_0^T (g_k(\cdot, u(r)) \eta'(u(r)), \psi(r, \cdot)) dW(r) \\ &+ \frac{1}{2} \int_0^T (G^2(\cdot, u(r)) \eta''(u(r, \cdot)), \psi(r, \cdot)) dr \geq 0, \end{aligned} \tag{2.1}$$

a.s., where $\Psi(u(r)) = \int_0^u a(\xi) \eta'(\xi) d\xi$ and $a(\xi) = A'(\xi)$.

Remark 2.1 *Comparing the above Definition 2.1 with Definition 2.1 in [1], we get that the above Definition 2.1 is the case "intermediate". It is well-known that "classical \Rightarrow entropy \Rightarrow intermediate \Rightarrow weak", see Remark 4.2 in [1].*

We cannot define the solution as the Definition 2.1 in [1], because we cannot prove the solutions of (1.1) belong to $BV(\mathbb{R}^d)$ even if the initial data belong to $BV(\mathbb{R}^d)$. When $\nu = 0$

in (1.1), Chen et al. [5] obtain the Fractional BV-estimate, see [5, Theorem 7]. Thus it is impossible to get the BV-estimate of solution to (1.1) under the assumption that the noise term depends on the spatial variable x . Besides, the BV-estimate of solution to (1.1) with $\Phi \equiv 0$ was obtained in [20]. That is, the deterministic nonlocal conservation law keeps the Bounded Variation property.

Remark 2.2 The solution defined in Definition 2.1 satisfies the initial condition in the following sense: for any compact set $K \subset \mathbb{R}^d$,

$$\operatorname{ess\,lim}_{t \rightarrow 0^+} \mathbb{E} \int_K |u - u_0| dx = 0.$$

The proof is exactly as that of Remark 2.7 in [2].

Inspired by [6], we give the following definitions.

Definition 2.2 (Kinetic measure) We say that a map m from Ω to the set of non-negative finite measure over $\mathbb{R}^d \times [0, T] \times \mathbb{R}$ is a kinetic measure if

1. m is measurable, in the sense that for each $\phi \in C_b(\mathbb{R}^d \times [0, T] \times \mathbb{R})$, $\langle m, \phi \rangle : \Omega \rightarrow \mathbb{R}$ is,
2. m vanishes for large ξ : if $B_R^c = \{\xi \in \mathbb{R}, |\xi| \geq R\}$, then

$$\lim_{R \rightarrow \infty} \mathbb{E} m(\mathbb{R}^d \times [0, T] \times B_R^c) = 0,$$

3. for all $\phi \in C_b(\mathbb{R}^{d+1})$, the process

$$t \mapsto \int_{\mathbb{R}^d \times [0, t] \times \mathbb{R}} \phi(x, \xi) dm(x, s, \xi)$$

is predictable.

Definition 2.3 (Solution) Let $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$. A measurable function $u : \mathbb{R}^d \times [0, T] \times \Omega \rightarrow \mathbb{R}$ is said to be a solution to (1.1) with initial datum u_0 if $\{u(t)\}$ is predictable, for all $p \geq 1$, there exists $C_p \geq 0$ such that

$$\mathbb{E} \left(\operatorname{ess\,sup}_{t \in [0, T]} \|u(t)\|_{L^p(\mathbb{R}^d)}^p \right) \leq C_p,$$

and if there exists a kinetic measure m such that $f := \mathbf{1}_{u > \xi}$ satisfies: for all $\varphi \in C_c^2(\mathbb{R}^d \times [0, T] \times \mathbb{R})$,

$$\begin{aligned} & \int_0^T \langle f(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\ &= \int_0^T \langle f(t), (-\Delta_x)^{\frac{\alpha}{2}} \varphi(t) \rangle dt - \sum_{k \geq 1} \int_0^T \int_{\mathbb{R}^d} g_k(x, u(x, t)) \varphi(x, t, u(x, t)) dx d\beta_k(t) \\ & \quad - \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \partial_\xi \varphi(x, t, u(x, t)) G^2(x, u(x, t)) dx dt + m(\partial_\xi \varphi), \end{aligned} \tag{2.2}$$

a.s., where $f_0(x, \xi) = \mathbf{1}_{u_0(x) > \xi}$, $G^2 := \sum_{k=1}^\infty |g_k|^2$ and $a(\xi) = A'(\xi)$.

In (2.2), we have used the brackets $\langle \cdot, \cdot \rangle$ to denote the duality between $C_c^\infty(\mathbb{R}^{d+1})$ and the space of distributions over \mathbb{R}^{d+1} . In what follows, we will denote similarly the integral

$$\langle F, G \rangle = \int_{\mathbb{R}^{d+1}} F(x, \xi) G(x, \xi) dx d\xi, \quad F \in L^p(\mathbb{R}^{d+1}), \quad G \in L^q(\mathbb{R}^{d+1}),$$

where $1 \leq p < \infty$ and q is the conjugate exponent of p . In (2.2), we also have indicated the dependence of g_k and G^2 on u , which is actually absent in the additive case and we have used the shorthand $m(\psi)$ for

$$m(\psi) = \int_{\mathbb{R}^d \times [0, T] \times \mathbb{R}} \psi(x, t, \xi) dm(x, t, \xi), \quad \psi \in C_c(\mathbb{R}^d \times [0, T] \times \mathbb{R}).$$

Equation (2.2) is the weak form of the equation

$$\left(\partial_t + a(\xi) \cdot \nabla_x + (-\Delta_x)^{\frac{\alpha}{2}} \right) \mathbf{1}_{u(x, t) > \xi} = \delta_{u=\xi} \Phi \dot{W} + \partial_\xi \left(m - \frac{1}{2} G^2 \delta_{u=\xi} \right). \quad (2.3)$$

Now, we present a formal derivation of equation (2.3) from (1.1) for regular solution which is similar to [6]. It is essentially a consequence of Itô formula. Indeed, by the identity $(\mathbf{1}_{u > \xi}, \theta') := \int_{\mathbb{R}} \mathbf{1}_{u > \xi} \theta' d\xi = \theta(u) - \theta(-\infty)$ for $\theta \in C^\infty(\mathbb{R})$, it yields that

$$\begin{aligned} d(\mathbf{1}_{u > \xi}, \theta') &= \theta'(u) \left(-a(u) \cdot \nabla u - (-\Delta_x)^{\frac{\alpha}{2}} u + \Phi(u) dW \right) + \frac{1}{2} \theta''(u) G^2 dt \\ &= -\operatorname{div} \left(\int^u a(\xi) \theta'(\xi) d\xi \right) dt + \frac{1}{2} \theta''(u) G^2 dt + \theta'(u) \Phi(u) dW \\ &\quad - (-\Delta_x)^{\frac{\alpha}{2}} \theta(u) - \theta'(u) (-\Delta_x)^{\frac{\alpha}{2}} u + (-\Delta_x)^{\frac{\alpha}{2}} \theta(u) \\ &= -\operatorname{div}(a \mathbf{1}_{u > \xi}, \theta') dt - \frac{1}{2} (\partial_\xi (G^2 \delta_{u=\xi}), \theta') + (\delta_{u=\xi} \Phi dW, \theta') \\ &\quad - ((-\Delta_x)^{\frac{\alpha}{2}} \mathbf{1}_{u > \xi}, \theta') + (\partial_\xi m, \theta'), \end{aligned}$$

where $(\partial_\xi m, \theta') = (-\Delta_x)^{\frac{\alpha}{2}} \theta(u) - \theta'(u) (-\Delta_x)^{\frac{\alpha}{2}} u$, we have used the following fact

$$\begin{aligned} (-\Delta_x)^{\frac{\alpha}{2}} \theta(u) &= C_d(\alpha) \int_{\mathbb{R}^d \setminus \{0\}} \frac{\theta(u) - \theta(u(x+z, t))}{|z|^{d+\alpha}} dz \\ &= C_d(\alpha) \int_{\mathbb{R}^d \setminus \{0\}} \frac{1}{|z|^{d+\alpha}} \left(\int_{u(x+z, t)}^u \theta'(\xi) d\xi \right) dz \\ &= C_d(\alpha) \int_{\mathbb{R}^d \setminus \{0\}} \frac{(\mathbf{1}_{u > \xi} - \mathbf{1}_{u(x+z, t) > \xi}, \theta'(\xi))}{|z|^{d+\alpha}} dz \\ &= ((-\Delta_x)^{\frac{\alpha}{2}} \mathbf{1}_{u > \xi}, \theta'). \end{aligned}$$

Next, we calculate m . Firstly, let $\theta \in C_c^2(\mathbb{R})$ be a convex function and we have

$$\langle \partial_\xi m, \theta' \rangle = -\langle m, \theta'' \rangle = - \int_{\mathbb{R}} m(x, t, \xi) \theta''(\xi) d\xi.$$

Assume that $\theta_\epsilon \in C_c^2(\mathbb{R})$ is a convex function satisfying $\lim_{\epsilon \rightarrow 0} \theta_\epsilon(\xi) = |v - \xi|$, $\lim_{\epsilon \rightarrow 0} \theta'_\epsilon(\xi) = \operatorname{sgn}(\xi - v)$ and $\lim_{\epsilon \rightarrow 0} \theta''_\epsilon(\xi) = \delta_{v=\xi}$, where $v \in \mathbb{R}$, then we get

$$\begin{aligned} m(x, t, v) &= - \lim_{\epsilon \rightarrow 0} \langle \partial_\xi m, \theta'_\epsilon \rangle \\ &= - \lim_{\epsilon \rightarrow 0} (-\Delta_x)^{\frac{\alpha}{2}} \theta_\epsilon(u) - \theta'_\epsilon(u) (-\Delta_x)^{\frac{\alpha}{2}} u \\ &= \operatorname{sgn}(u - v) (-\Delta_x)^{\frac{\alpha}{2}} u - (-\Delta_x)^{\frac{\alpha}{2}} |u - v|. \end{aligned}$$

From the definition of $(-\Delta)^{\frac{\alpha}{2}}$, we have another representation for the kinetic measure m . It follows from Lemma 2.1 that

$$\begin{aligned}
(\partial_\xi m, \theta') &= (-\Delta_x)^{\frac{\alpha}{2}} \theta(u) - \theta'(u) (-\Delta_x)^{\frac{\alpha}{2}} u \\
&= C_d(\alpha) \int_{\mathbb{R}^d \setminus \{0\}} \frac{1}{|z|^{d+\alpha}} [\theta(u) - \theta(u(x+z, t)) - \theta'(u)(u - u(x+z, t))] dz \\
&= \frac{C_d(\alpha)}{2} \int_{\mathbb{R}^d \setminus \{0\}} \frac{1}{|z|^{d+\alpha}} \theta''((1-\tau)u + \tau u(x+z, t)) (u - u(x+z, t))^2 dz \\
&= -\frac{C_d(\alpha)}{2} \int_{\mathbb{R}^d \setminus \{0\}} \frac{1}{|z|^{d+\alpha}} (\partial_\xi((u - u(x+z, t))^2 \delta_{\xi=(1-\tau)u+\tau u(x+z, t)}, \theta'(\xi)) dz,
\end{aligned}$$

which implies that

$$m(x, t, \xi) = \frac{C_d(\alpha)}{2} \int_{\mathbb{R}^d \setminus \{0\}} \frac{(u - u(x, t+z))^2}{|z|^{d+\alpha}} \delta_{\xi=(1-\tau)u+\tau u(x+z, t)} dz, \quad \tau \in (0, 1) \quad (2.4)$$

The above representation shows that m_1 is a non-negative measure.

Taking $\theta(\xi) = \int_{-\infty}^{\xi} \varphi$, we then obtain the formulation. Besides the nonnegative measure given by (2.4), the kinetic measure m described in (2.2) contains another non-negative measure, which is sometimes interpreted as a Lagrange multiplier for the evolution of f by $\partial_t + a \cdot \nabla$ under the constraint $f = \text{graph} = \mathbf{1}_{u>\xi}$. It will be arose when u becomes discontinuous. Indeed, if one we add the viscosity term $\varepsilon \Delta u$ in equation (1.1), then the measure m can be written as

$$\begin{aligned}
m(\phi) &= \varepsilon \int_{\mathbb{R}^d \times [0, T]} \phi(x, t, u(x, t)) |\nabla u|^2 dx dt \\
&\quad + \int_{\mathbb{R}^d \times [0, T] \times \mathbb{R}} \phi(x, t, \xi) \frac{C_d(\alpha)}{2} \int_{\mathbb{R}^d \setminus \{0\}} \frac{(u - u(x, t+z))^2}{|z|^{d+\alpha}} \delta_{\xi=(1-\tau)u+\tau u(x+z, t)} dz dx dt d\xi.
\end{aligned}$$

Now, we are in a position to show the relationship between entropy solutions and kinetic solutions for (1.1)- (1.2).

Theorem 2.1 (Kinetic formulation) *Let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$. For a measurable function $u : \mathbb{R}^d \times [0, T] \times \Omega \rightarrow \mathbb{R}$, it is equivalent to be a kinetic solution to (1.1), i.e. both the solutions in sense of Definitions 2.1 and 2.3 are equivalent.*

Proof. Choosing test function $\varphi(x, t, \xi) = \psi(x, t) \eta'(\xi)$ in (2.2) and noting that η is a convex function, we have

$$\begin{aligned}
\langle f(t), (-\Delta_x)^{\frac{\alpha}{2}} \varphi(t) \rangle &= \langle (-\Delta_x)^{\frac{\alpha}{2}} f(t), \psi(x, t) \eta'(\xi) \rangle \\
&= \langle (-\Delta_x)^{\frac{\alpha}{2}} \eta(u(t)), \psi(x, t) \rangle \\
&= \langle \eta(u(t)), (-\Delta_x)^{\frac{\alpha}{2}} \psi(x, t) \rangle.
\end{aligned}$$

Using the above inequality and the facts $m(\eta'') \geq 0$ and $n(\eta'') \geq 0$, (2.2) implies the inequality in Definition 2.1. That is, a kinetic solution will be a entropy solution.

Conversely, similar to [6], one defines the measure m by

$$m(\eta'' \psi) = \int_0^T (\eta(u), \partial_t \psi) dr + (\eta(u_0), \psi(0)) + \int_0^T (\Psi(u), \nabla \psi) dr$$

$$\begin{aligned}
& + \sum_{k \geq 1} \int_0^T (g_k(\cdot, u(r)) \eta'(u(r)), \psi) d\beta_k(r) + \frac{1}{2} \int_0^T (G^2(\cdot, u(r)) \eta''(u(r)), \psi) dr \\
& - \int_0^T \int_{\mathbb{R}^d} \eta(u(r)) (-\Delta)^{\frac{\alpha}{2}} \psi dx dr,
\end{aligned}$$

then one derives (2.1). Moreover, by virtue of above representation of m , we prove that m is a kinetic measure. \square

In order to prove the existence of solution, we introduce the following definitions, see [6].

Definition 2.4 (Young measure) *Let (X, λ) be a finite measure space. Let $\mathcal{P}_1(\mathbb{R})$ denote the set of probability measures on \mathbb{R} . We say that a map $\nu : X \rightarrow \mathcal{P}_1(\mathbb{R})$ is a Young measure on X if, for all $\phi \in C_b(\mathbb{R})$, the map $z \mapsto \nu_z(\phi)$ from X to \mathbb{R} is measurable. We say that a Young measure ν vanishes at infinity if, for every $p \geq 1$,*

$$\int_X \int_{\mathbb{R}} |\xi|^p d\nu_z(\xi) d\lambda(z) < +\infty. \quad (2.5)$$

Definition 2.5 (Kinetic function) *Let (X, λ) be a finite measure space. A measurable function $f : X \times \mathbb{R} \rightarrow [0, 1]$ is said to be a kinetic function if there exists a Young measure ν on X that vanishes at infinity such that, for λ -a.e. $z \in X$, for all $\xi \in \mathbb{R}$,*

$$f(z, \xi) = \nu_z(\xi, +\infty).$$

We say that f is an equilibrium if there exists a measurable function $u : X \rightarrow \mathbb{R}$ such that $f(z, \xi) := \mathbf{1}_{u(z) > \xi}$ a.e., or, equivalently, $\nu_z = \delta_{u(z)}$ for a.e. $z \in X$.

If $f : X \times \mathbb{R} \rightarrow [0, 1]$ is a kinetic function, we denote by \bar{f} the conjugate function $\bar{f} = 1 - f$. We can define the kinetic function in another way (see [14])

$$\chi_u(\xi) = \mathbf{1}_{(0, u(x, t))}(\xi) - \mathbf{1}_{(u(x, t), 0)}(\xi) = \mathbf{1}_{u > \xi} - \mathbf{1}_{0 > \xi},$$

which is decreasing faster than any power of ξ at infinity. Contrary to f , $\chi_u(\xi)$ is integrable. Now, we recall the compactness of Young measures, see [6] for the proof.

Proposition 2.1 [6, Theorem 5] *Let (X, λ) be a finite measure space such that $L^1(X)$ is separable. Let (ν^n) be a sequence of Young measures on X satisfying (2.5) uniformly for some $p \geq 1$:*

$$\sup_n \int_X \int_{\mathbb{R}} |\xi|^p d\nu_z^n(\xi) d\lambda(z) < +\infty. \quad (2.6)$$

Then there exists a Young measure ν on X and a subsequence still denoted (ν^n) such that, for all $h \in L^1(X)$, for all $\phi \in C_b(\mathbb{R})$,

$$\lim_{n \rightarrow \infty} \int_X h(z) \int_{\mathbb{R}} \phi(\xi) d\nu_z^n d\lambda(z) = \int_X h(z) \int_{\mathbb{R}} \phi(\xi) d\nu_z d\lambda(z).$$

By Proposition 2.1, we have the following result. Let (f_n) be a sequence of kinetic functions on $X \times \mathbb{R}$: $f_n(z, \xi) = \nu_z^n(\xi, +\infty)$, where ν^n are Young measures on X satisfying (2.6). Let f be a kinetic function on $X \times \mathbb{R}$ such that $f_n \rightharpoonup f$ in $L^\infty(X \times \mathbb{R})$ weak-*. Assume that f_n and f are equilibria:

$$f_n(z, \xi) = \mathbf{1}_{u_n(z) > \xi}, \quad f(z, \xi) = \mathbf{1}_{u(z) > \xi}.$$

Then, for all $1 \leq q < p$, $u_n \rightarrow u$ in $L^q(X)$ strong.

Definition 2.6 (Generalized solution) *Let $f_0 : \Omega \times \mathbb{R}^{d+1} \mapsto [0, 1]$ be a kinetic function. A measurable function $f : \Omega \times \mathbb{R}^d \times [0, T] \times \mathbb{R} \mapsto [0, 1]$ is said to be a generalized solution to (1.1) with initial datum f_0 if $\{f(t)\}$ is predictable and is a kinetic function such that: for all $p \geq 1$, $\nu := -\partial_\xi f$ satisfies*

$$\mathbb{E} \left(\operatorname{ess\,sup}_{t \in [0, T]} \int_{\mathbb{R}^{d+1}} |\xi|^p d\nu_{x,t} dx \right) \leq C_p, \quad (2.7)$$

where C_p is a positive constant and: there exists a kinetic measure m such that for all $\varphi \in C_c^2(\mathbb{R}^d \times [0, T] \times \mathbb{R})$,

$$\begin{aligned} & \int_0^T \langle f(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\ &= \int_0^T \langle f(t), (-\Delta_x)^{\frac{\alpha}{2}} \varphi(t) \rangle dt - \sum_{k \geq 1} \int_0^T \int_{\mathbb{R}^{d+1}} g_k(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}(\xi) dx d\beta_k(t) \\ & \quad - \frac{1}{2} \int_0^T \int_{\mathbb{R}^{d+1}} \partial_\xi \varphi(x, t, \xi) G^2(x, \xi) d\nu_{x,t}(\xi) dx dt + m(\partial_\xi \varphi), \quad \text{a.s.} \end{aligned} \quad (2.8)$$

Note that the generalized solution (Definition 2.6) implies the solution (Definition 2.3). Indeed, if f is a generalized solution such that $f = \mathbf{1}_{u > \xi}$, then $u(x, t) = \int_{\mathbb{R}} (f - \mathbf{1}_{0 > \xi}) d\xi$, hence u is predictable. Moreover, if $\nu_{x,t}(\xi) = \delta_{u=\xi}$, then equality (2.8) implies (2.2).

Following [6], we shall show that any generalized solution admits possibly different left and right weak limits at any point $t \in [0, T]$ almost surely. This property is important to prove a comparison principle which allows to prove uniqueness. Meanwhile, it allows us to rewrite (2.8) in some stronger sense.

Proposition 2.2 (left and right weak limits) *Let f_0 be a kinetic initial datum. Let f be a generalized solution to (1.1) with initial datum f_0 . Then f admits almost surely left and right limits at all points $t_* \in [0, T]$. More precisely, for all $t_* \in [0, T]$ there exists some kinetic functions $f^{*,\pm}$ on $\Omega \times \mathbb{R}^{d+1}$ such that \mathbb{P} -a.s.*

$$\langle f(t_* - \varepsilon), \varphi \rangle \rightarrow \langle f^{*, -}, \varphi \rangle$$

and

$$\langle f(t_* + \varepsilon), \varphi \rangle \rightarrow \langle f^{*, +}, \varphi \rangle$$

as $\varepsilon \rightarrow 0$ for all $\varphi \in C_c^2(\mathbb{R}^{d+1})$. Moreover, almost surely,

$$\langle f^{*, +} - f^{*, -}, \varphi \rangle = - \int_{\mathbb{R}^d \times [0, T] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{1}_{\{t_*\}}(t) dm(x, t, \xi). \quad (2.9)$$

In particular, almost surely, the set of $t_* \in [0, T]$ such that $f^{*, +} \neq f^{*, -}$ is countable.

Proof. Following [6], for all $\varphi \in C_c^2(\mathbb{R}^{d+1})$, a.s., the map

$$\begin{aligned} J_\varphi : t &\rightarrow \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle ds - \int_0^t \langle f(s), (-\Delta_x)^{\frac{\alpha}{2}} \varphi \rangle ds \\ &\quad + \sum_{k \geq 1} \int_0^t \int_{\mathbb{R}^{d+1}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}(\xi) dx d\beta_k(s) \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^{d+1}} \partial_\xi \varphi(x, \xi) G^2(x, \xi) d\nu_{x,s}(\xi) dx ds \end{aligned}$$

is continuous on $[0, T]$. Taking the test function of the form $(x, t, \xi) \mapsto \varphi(x, \xi)\gamma(t)$, $\gamma \in C_c^1([0, T])$, $\varphi \in C_c^2(\mathbb{R}^{d+1})$, we get by using Fubini Theorem and the weak formulation (2.8)

$$\int_0^T \mathcal{J}_\varphi(t) \gamma'(t) dt + \langle f_0, \varphi \rangle \gamma(0) = \langle m, \partial_\xi \varphi \rangle(\gamma),$$

where $\mathcal{J}_\varphi(t) := \langle f(t), \varphi \rangle - J_\varphi(t)$. This shows that $\partial_t \mathcal{J}_\varphi$ is a measure on $(0, T)$, i.e., the function $\mathcal{J}_\varphi \in BV(0, T)$. Hence it admits left and right limits at all points $t_* \in [0, T]$. Since J_φ is continuous, this also holds for $\langle f(t), \varphi \rangle$: for all $t_* \in [0, T]$, the limits

$$\langle f, \varphi \rangle(t_*+) := \lim_{t \downarrow t_*} \langle f, \varphi \rangle(t) \quad \text{and} \quad \langle f, \varphi \rangle(t_*-) := \lim_{t \uparrow t_*} \langle f, \varphi \rangle(t)$$

exist. Then following the proof of Proposition 8 of [6], it is easy to complete the proof. \square

Using the Proposition 2.2, we can derive a kinetic formulation at given t . Taking a test function of the form $(x, t, \xi) \mapsto [K(T - t, \cdot) * \varphi(\cdot, \xi)](x) \gamma(t)$ where the kernel function K satisfies $K_t + (-\Delta)^{\frac{\alpha}{2}} K = 0$, and γ is the function

$$\gamma(s) = \begin{cases} 1, & s \leq t, \\ 1 - \frac{s-t}{\varepsilon}, & t \leq s \leq t + \varepsilon, \\ 0, & t + \varepsilon \leq s, \end{cases}$$

we obtain at the limit $[\varepsilon \rightarrow 0]$: for all $t \in [0, T]$ and $\varphi \in C_c^2(\mathbb{R}^{d+1})$,

$$\begin{aligned} & -\langle f^+(t), \tilde{\varphi}(t) \rangle + \langle f_0, \tilde{\varphi}(0) \rangle + \int_0^t \langle f(s), a(\xi) \cdot \nabla \tilde{\varphi} \rangle ds \\ &= -\sum_{k \geq 1} \int_0^t \int_{\mathbb{R}^{d+1}} g_k(x, \xi) \tilde{\varphi}(x, \xi) d\nu_{x,s}(\xi) dx d\beta_k(s) \\ & \quad - \frac{1}{2} \int_0^t \int_{\mathbb{R}^{d+1}} \partial_\xi \tilde{\varphi}(x, \xi) G^2(x, \xi) d\nu_{x,s}(\xi) dx ds + \langle m, \partial_\xi \tilde{\varphi} \rangle([0, t]), \quad a.s., \end{aligned} \quad (2.10)$$

where

$$\tilde{\varphi}(x, t) = [K(T - t, \cdot) * \varphi(\cdot, \xi)](x), \quad \langle m, \partial_\xi \tilde{\varphi} \rangle([0, t]) = \int_{\mathbb{R}^d \times [0, t] \times \mathbb{R}} \partial_\xi \tilde{\varphi}(x, \xi) dm(x, s, \xi).$$

We remark that if $\varphi \in C_c^2(\mathbb{R}^{d+1})$, then $(-\Delta)^{\frac{\alpha}{2}} \varphi$ makes sense and also for $(-\Delta)^{\frac{\alpha}{2}} \tilde{\varphi}$. Using the following fact

$$\begin{aligned} \langle f^+(t), \tilde{\varphi}(t) \rangle &= \int_{\mathbb{R}^{d+1}} f^+(x, t, \xi) \tilde{\varphi}(x, t, \xi) dx d\xi \\ &= \int_{\mathbb{R}^{d+1}} f^+(x, t, \xi) \int_{\mathbb{R}^d} K(T - t, y) \varphi(x - y, \xi) dy dx d\xi \\ &= \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^d} K(T - t, y) f^+(x, t - y, \xi) dy \varphi(x, \xi) dx d\xi \\ &= \langle \tilde{f}^+(t), \varphi \rangle, \end{aligned}$$

we can rewrite the equality (2.10), that is, the convolution of K and φ can be changed into the convolution of K and another function.

Remark 2.3 (The case of equilibrium) Suppose that $f^{*, -}$ is at equilibrium in (2.9): there is a random variable $u^* \in L^1(\mathbb{R}^d)$ so that $f^{*, -} = \mathbf{1}_{u^* > \xi}$ a.s. Let m^* denote the restriction of m to $\mathbb{R}^d \times \{t_*\} \times \mathbb{R}$. We thus have

$$f^{*, +} - \mathbf{1}_{u^* > \xi} = \partial_\xi m^*.$$

By the condition 2 in Definition 2.2, one achieves that

$$\int_{\mathbb{R}} (f^{*, +}(x, \xi) - \mathbf{1}_{0 > \xi}) d\xi = \int_{\mathbb{R}} (\mathbf{1}_{u^* > \xi} - \mathbf{1}_{0 > \xi}) d\xi = u^*.$$

Observing that

$$p^* : \xi \rightarrow \int_{-\infty}^{\xi} (\mathbf{1}_{u^* > \zeta} - f^{*, +}(\zeta))$$

is non-negative and $\partial_\xi(m^* + p^*) = 0$, thus $m^* + p^*$ is constant and actually vanishes by the condition 2 in Definition 2.2 and the obvious fact that p^* also vanishes when $\xi \rightarrow \infty$. Since $m^* \geq 0$, we conclude $m^* = 0$, which suggests $f^{*, +} = f^{*, -}$.

3 Uniqueness and existence of kinetic solutions

In this section, we are interested in the Cauchy problem (1.1)-(1.2) and it is ready for us to state our main result.

Theorem 3.1 Let (1.3) and (1.4) hold. Then there is a unique kinetic solution of the nonlocal Cauchy problem (1.1)-(1.2).

We will use the doubling variables method to prove the uniqueness. Let f_i ($i = 1, 2$) be the generalized solution of the equation

$$\partial_t u_i(x, t) + (-\Delta_x)^{\frac{\alpha}{2}} u_i + \operatorname{div}(A(u_i)) = \Phi(u_i) \partial_t W(t), \quad x \in \mathbb{R}^d, \quad t > 0. \quad (3.1)$$

Set $\bar{f} = 1 - f$. In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.1 Let f_i , $i = 1, 2$, be generalized solution to (3.1). Then, for $0 \leq t \leq T$, and non-negative functions $\rho \in C_c^\infty(\mathbb{R}^d)$ and $\psi \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \check{\rho}(x - y) \psi(\xi - \zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta dx dy \\ & \leq \mathbb{E} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \check{\rho}(x - y) \psi(\xi - \zeta) f_{1,0}^\pm(x, \xi) \bar{f}_{2,0}^\pm(y, \zeta) d\xi d\zeta dx dy + I_{\check{\rho}} + I_\psi, \end{aligned} \quad (3.2)$$

where ,

$$\begin{aligned} \check{\rho}(x) &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K(T - t, z_2) K(T - t, z_1 + z_2) dz_2 \right) \rho(x - z_1) dz_1, \\ I_{\check{\rho}} &= \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) (a(\xi) - a(\zeta)) \psi(\xi - \zeta) d\xi d\zeta \cdot \nabla_x \check{\rho}(x - y) dx dy ds, \end{aligned}$$

and

$$I_\psi = \frac{1}{2} \int_{\mathbb{R}^{2d}} \check{\rho}(x - y) \mathbb{E} \int_0^t \int_{\mathbb{R}^2} \psi(\xi - \zeta) \sum_{k \geq 1} |g_k(x, \xi) - g_k(x, \zeta)|^2 d\nu_{x, \xi}^1 \otimes d\nu_{x, \xi}^2 dx dy ds.$$

Proof. Since both ρ and ψ have compact support, it is easy to check each term in (3.2) is finite. Set $G_1^2(x, \xi) = \sum_{k=1}^{\infty} |g_k(x, \xi)|^2$ and $G_2^2(x, \zeta) = \sum_{k=1}^{\infty} |g_k(x, \zeta)|^2$.

Let $\varphi_1(x, \xi) \in C_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi)$ and $\varphi_2(x, \zeta) \in C_c^\infty(\mathbb{R}_y^d \times \mathbb{R}_\zeta)$. Recall that

$$\begin{aligned}\tilde{\varphi}_1(x, \xi) &= \int_{\mathbb{R}^d} K(T - x, t - z) \varphi_1(z, \xi) dz, \\ \tilde{\varphi}_2(y, \zeta) &= \int_{\mathbb{R}^d} K(T - t, y - z) \varphi_2(z, \zeta) dz.\end{aligned}$$

By (2.10), we have

$$\langle f_1^+(t), \tilde{\varphi}_1 \rangle = \langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle([0, t]) + F_1(t),$$

where

$$F_1(t) = \sum_{k \geq 1} \int_0^t \int_{\mathbb{R}^{d+1}} g_{k,1} \tilde{\varphi}_1 d\nu_{x,s}^1(\xi) dx d\beta_k(s)$$

and

$$\begin{aligned}\langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle([0, t]) &= \langle f_{1,0}, \tilde{\varphi}_1 \rangle + \int_0^t \langle f_1, a \cdot \nabla_x \tilde{\varphi}_1 \rangle ds \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^{d+1}} \partial_\xi \tilde{\varphi}_1(x, \xi) G_1^2(x, \xi) d\nu_{x,s}^1(\xi) dx ds - \langle m_1, \partial_\xi \tilde{\varphi}_1 \rangle([0, t]).\end{aligned}$$

Using Remark 2.3, $\langle m_1, \partial_\xi \tilde{\varphi}_1 \rangle(\{0\}) = 0$ and thus $\langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle(\{0\})$ is $\langle f_{1,0}, \tilde{\varphi}_1 \rangle$. Similarly, we have

$$\langle \bar{f}_2^+(t), \tilde{\varphi}_2 \rangle = \langle \bar{m}_2^*, \partial_\zeta \tilde{\varphi}_2 \rangle([0, t]) + \bar{F}_2(t),$$

where

$$\bar{F}_2(t) = - \sum_{k \geq 1} \int_0^t \int_{\mathbb{R}^{d+1}} g_{k,2} \tilde{\varphi}_2 d\nu_{y,s}^2(\zeta) dy d\beta_k(s)$$

and

$$\begin{aligned}\langle \bar{m}_2^*, \partial_\zeta \tilde{\varphi}_2 \rangle([0, t]) &= \langle \bar{f}_{2,0}, \tilde{\varphi}_2 \rangle + \int_0^t \langle \bar{f}_2, a \cdot \nabla_y \tilde{\varphi}_2 \rangle ds \\ &\quad - \frac{1}{2} \int_0^t \int_{\mathbb{R}^{d+1}} \partial_\zeta \tilde{\varphi}_2(y, \zeta) G_2^2(y, \zeta) d\nu_{y,s}^2(\zeta) dy ds + \langle m_2, \partial_\zeta \tilde{\varphi}_2 \rangle([0, t]),\end{aligned}$$

where $\langle \bar{m}_2^*, \partial_\zeta \tilde{\varphi}_2 \rangle(\{0\}) = \langle \bar{f}_{2,0}, \tilde{\varphi}_2 \rangle$. Integrating by parts for functions of finite variation

$$\langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle([0, t]) \langle \bar{m}_2^*, \partial_\zeta \tilde{\varphi}_2 \rangle([0, t]),$$

we get

$$\begin{aligned}\langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle([0, t]) \langle \bar{m}_2^*, \partial_\zeta \tilde{\varphi}_2 \rangle([0, t]) &= \langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle(\{0\}) \langle \bar{m}_2^*, \partial_\zeta \tilde{\varphi}_2 \rangle(\{0\}) \\ &\quad + \int_{(0,t]} \langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle([0, s]) d\langle \bar{m}_2^*, \partial_\zeta \tilde{\varphi}_2 \rangle(s) \\ &\quad + \int_{(0,t]} \langle \bar{m}_2^*, \partial_\zeta \tilde{\varphi}_2 \rangle([0, s]) d\langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle(s).\end{aligned}$$

Since \bar{F}_2 is continuous and $\bar{F}_2(0) = 0$, we have

$$\langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle([0, t]) \bar{F}_2(t) = \int_0^t \langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle([0, s]) d\bar{F}_2(s) + \int_0^t \bar{F}_2(s) \langle m_1^*, \partial_\xi \tilde{\varphi}_1 \rangle(ds).$$

Denote $\tilde{\sigma} = \tilde{\varphi}_1 \tilde{\varphi}_2$. Using Itô formula for $F_1(t) \bar{F}_2(t)$, we obtain that

$$\langle f_1^+(t), \tilde{\varphi}_1 \rangle \langle \bar{f}_2^+(t), \tilde{\varphi}_2 \rangle = \langle \langle f_1^+(t) \bar{f}_2^+(t), \tilde{\sigma} \rangle \rangle,$$

where

$$\begin{aligned} & \mathbb{E} \langle \langle f_1^+(t) \bar{f}_2^+(t), \tilde{\sigma} \rangle \rangle - \langle \langle f_{1,0}^+ \bar{f}_{2,0}^+, \tilde{\sigma}_0 \rangle \rangle \\ = & \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} f_1 \bar{f}_2 [a(\xi) \cdot \nabla_x + a(\zeta) \cdot \nabla_y] \tilde{\sigma} d\xi d\zeta dx dy ds \\ & \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} f_1 \bar{f}_2 [(-\Delta_x)^{\frac{\alpha}{2}} + (-\Delta_y)^{\frac{\alpha}{2}}] \tilde{\sigma} d\xi d\zeta dx dy ds \\ & + \frac{1}{2} \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \partial_\xi \tilde{\sigma} \bar{f}_2(s) G_1^2 d\nu_{(x,s)}^1(\xi) d\zeta dx dy ds \\ & - \frac{1}{2} \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \partial_\zeta \tilde{\sigma} f_1(s) G_2^2 d\nu_{(y,s)}^2(\zeta) d\xi dy dx ds \\ & - \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} G_{1,2} \tilde{\sigma} d\nu_{(x,s)}^1(\xi) d\nu_{(y,s)}^2(\zeta) dx dy \\ & - \mathbb{E} \int_{(0,t]} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \bar{f}_2^+ \partial_\xi \tilde{\sigma} dm_1(x, s, \xi) d\zeta dy \\ & + \mathbb{E} \int_{(0,t]} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} f_1^- \partial_\zeta \tilde{\sigma} dm_2(y, s, \zeta) d\xi dx, \end{aligned} \tag{3.3}$$

where $G_{1,2}(x, y; \xi, \zeta) := \sum_{k \geq 1} g_{k,1}(x, \xi) g_{k,2}(y, \zeta)$ and $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the duality distribution over $\mathbb{R}_x^d \times \mathbb{R}_\xi^d \times \mathbb{R}_y^d \times \mathbb{R}$, and $\tilde{\sigma}_0 = \tilde{\sigma}|_{t=0}$. By a density argument, (3.3) remains true for any test function $\sigma \in C_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d \times \mathbb{R}_y^d \times \mathbb{R})$. Let $\sigma = \rho\psi$, where $\rho = \rho(x - y)$ and $\psi = \psi(\xi - \zeta)$. By the definition of $\tilde{\varphi}_i$, $i = 1, 2$, we have

$$\begin{aligned} \tilde{\sigma} &= \int_{\mathbb{R}^d} K(T - t, z_2) \varphi_2(y - z_2, \zeta) dz_2 \int_{\mathbb{R}^d} K(T - t, z_1) \varphi_1(x - z_1, \xi) dz_1 \\ &= \int_{\mathbb{R}^d} K(T - t, z_2) \int_{\mathbb{R}^d} K(T - t, z_1) \varphi_1(x - z_1, \xi) \varphi_2(y - z_2, \zeta) dz_1 dz_2 \\ &= \psi(\xi - \zeta) \int_{\mathbb{R}^d} K(T - t, z_2) \int_{\mathbb{R}^d} K(T - t, z_1) \rho(x - y - z_1 + z_2) dz_1 dz_2 \\ &= \psi(\xi - \zeta) \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K(T - t, z_2) K(T - t, z_1 + z_2) dz_2 \right) \rho(x - y - z_1) dz_1 \\ &=: \psi(\xi - \zeta) \check{\rho}(x - y). \end{aligned}$$

Noting that

$$(\nabla_x + \nabla_y) \tilde{\sigma} = 0, \quad (\partial_\xi + \partial_\zeta) \tilde{\sigma} = 0,$$

we have

$$\begin{aligned} & \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} f_1 \bar{f}_2 [a(\xi) \cdot \nabla_x + a(\zeta) \cdot \nabla_y] \tilde{\sigma} d\xi d\zeta dx dy ds \\ &= \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) (a(\xi) - a(\zeta)) \psi(\xi - \zeta) d\xi d\zeta \cdot \nabla_x \check{\rho}(x - y) dx dy ds. \end{aligned}$$

The last term in (3.3) is

$$\begin{aligned} & \mathbb{E} \int_{(0,t]} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} f_1^- \partial_\zeta \tilde{\sigma} dm_2(y, s, \zeta) d\xi dx \\ &= -\mathbb{E} \int_{(0,t]} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} f_1^- \partial_\xi \tilde{\sigma} dm_2(y, s, \zeta) d\xi dx \\ &= -\mathbb{E} \int_{(0,t]} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \tilde{\sigma} d\nu_{(x,s)}^{1,-} dm_2(y, s, \zeta) d\xi dx \\ &\leq 0 \end{aligned}$$

since $\alpha \geq 0$ and m_2, n_2 are non-negative measure. Similarly, we have

$$-\mathbb{E} \int_{(0,t]} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \bar{f}_2^+ \partial_\xi \tilde{\sigma} dm_1(x, s, \xi) d\zeta dy = -\mathbb{E} \int_{(0,t]} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \tilde{\sigma} d\nu_{(y,s)}^{2,+} dm_1(x, s, \xi) d\zeta dy \leq 0.$$

Integrating by part, we get

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \partial_\xi \tilde{\sigma} \bar{f}_2(s) G_1^2 d\nu_{(x,s)}^1(\xi) d\zeta dx dy ds \\ & - \frac{1}{2} \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \partial_\zeta \tilde{\sigma} f_1(s) G_2^2 d\nu_{(y,s)}^2(\zeta) d\xi dy dx ds \\ & - \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} G_{1,2} \tilde{\sigma} d\nu_{(x,s)}^1(\xi) d\nu_{(y,s)}^2(\zeta) dx dy \\ &= \frac{1}{2} \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \tilde{\sigma} (G_1^2 - 2G_{1,2} + G_2^2) d\nu_{(x,s)}^1 \otimes d\nu_{(y,s)}^2(\xi, \zeta) dx dy ds \\ &= \frac{1}{2} \int_{\mathbb{R}^{2d}} \check{\rho}(x - y) \mathbb{E} \int_0^t \int_{\mathbb{R}^2} \psi(\xi - \zeta) \\ & \quad \times \sum_{k \geq 1} |g_k(x, \xi) - g_k(x, \zeta)|^2 d\nu_{x,\xi}^1 \otimes d\nu_{x,\xi}^2 dx dy ds \\ &=: I_\psi. \end{aligned}$$

Combining the above discussion, we obtain the desired results. \square

Proof of Theorem 3.1 We first use the Lemma 3.1 to prove the uniqueness. The additive case: $\Phi(u)$ independent on u . Let $f_i, i = 1, 2$ be two generalized solution to (1.1). Then, we use (3.2) with g_k independent of ξ and ζ . By (1.3) the last term I_ψ is bounded by

$$\frac{tD_1}{2} \|\psi\|_{L^\infty} \int_{\mathbb{R}^{2d}} |x - y|^2 \check{\rho}(x - y) dx dy.$$

Note that if $\rho(x) \equiv C$, by using the properties of the heat kernel K , we have

$$\check{\rho}(x) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K(T - t, z_2) K(T - t, z_1 + z_2) dz_2 \right) \rho(x - z_1) dz_1$$

$$\begin{aligned}
&= C \int_{\mathbb{R}^d} K(T-t, z_2) \left(\int_{\mathbb{R}^d} K(T-t, z_1+z_2) dz_1 \right) dz_2 \\
&= C.
\end{aligned}$$

Taking $\psi := \psi_\delta$ and $\rho = \rho_\epsilon$, where (ψ_δ) and (ρ_ϵ) are approximations to the identity on \mathbb{R} and \mathbb{R}^d respectively, we obtain

$$I_\psi \leq \frac{t\tilde{D}_1}{2} \epsilon^2 \delta^{-1}, \quad (3.4)$$

where $\tilde{D}_1 = D_1 \int_{\mathbb{R}^d} z^2 \rho(z) dz < \infty$ because of the compact support of ρ . Let $t \in [0, T]$, $t_n \downarrow t$ and $\nu_{x,t}^{i,+}$ be a weak-limit of $\nu_{x,t_n}^{i,+}$ in sense of (2.6). Then $\nu_{x,t}^{i,+}$ satisfies

$$\mathbb{E} \int_{\mathbb{R}^d} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}^{i,+} dx \leq C_p,$$

and we have a similar bound for $\nu^{i,-}$. Denote

$$\tilde{K}(T-x, t) = \int_{\mathbb{R}^d} K(T-t, z) K(T-x, t+z) dz.$$

We can rewrite the integration as

$$\begin{aligned}
&\mathbb{E} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \tilde{K}(T-x, t-y) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(x, t, \xi) d\xi dy dx \\
&= \mathbb{E} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \check{\rho}_\epsilon(x-y) \psi_\delta(\xi-\zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta dx dy + \eta_t(\epsilon, \delta),
\end{aligned}$$

where $\lim_{\epsilon, \delta \rightarrow 0} \eta_t(\epsilon, \delta) = 0$. Now, we need a bound on the term I_ρ . Since a has at most polynomial growth, similar to the proof of [6, Theorem 11, pp 1029], there exists a positive constant C_p such that

$$\begin{aligned}
&\left| \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) (a(\xi) - a(\zeta)) \psi_\delta(\xi - \zeta) d\xi d\zeta \right. \\
&\quad \left. \cdot \nabla_x \check{\rho}_\epsilon(x-y) dx dy ds \right| \leq t C_p \delta \epsilon^{-1}.
\end{aligned} \quad (3.5)$$

We then gather (3.4), (3.5) and (3.2) to deduce for $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \tilde{K}(T-x, t-y) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(x, t, \xi) d\xi dy dx \\
&\leq \mathbb{E} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \tilde{K}(T, x-y) f_{1,0} \bar{f}_{2,0} d\xi dy dx + r(\epsilon, \delta),
\end{aligned}$$

where the remainder $r(\epsilon, \delta)$ is

$$r(\epsilon, \delta) = T C_p \delta \epsilon^{-1} + \frac{T\tilde{D}_1}{2} \epsilon^2 \delta^{-1} + \eta_t(\epsilon, \delta) + \eta_0(\epsilon, \delta).$$

Taking $\delta = \epsilon^{\frac{4}{3}}$ and letting $\epsilon \rightarrow 0$, we have $\lim_{\epsilon \rightarrow 0} r(\epsilon, \delta) = 0$ and

$$\begin{aligned}
&\mathbb{E} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \tilde{K}(T-x, t-y) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(x, t, \xi) d\xi dy dx \\
&\leq \mathbb{E} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \tilde{K}(T, x-y) f_{1,0} \bar{f}_{2,0} d\xi dy dx.
\end{aligned} \quad (3.6)$$

Assume that f is a generalized solution to (1.1) with initial datum $\mathbf{1}_{u_0 > \xi}$. Since f_0 is the Heaviside function, we get the identity $f_0 \bar{f}_0 = 0$. Taking $f_1 = f_2 = f$, by the positive property of K ($K(x, t) > 0$ for any $t > 0$ and $x \in \mathbb{R}^d$), we deduce that $f^+(1 - f^+) = 0$ a.e., i.e. $f^+ \in \{0, 1\}$ a.e.. The fact $-\partial_\xi f^+$ is a Young measure gives the conclusion: indeed, by Fubini Theorem, for any $t \in [0, T]$, there is a set E_t of full measure in $\mathbb{R}^d \times \Omega$ such that, for $(x, \omega) \in E_t$, $f^+(x, t, \xi, \omega) \in \{0, 1\}$ for a.e. $\xi \in \mathbb{R}$. Recall that $-\partial_\xi f^+(x, t, \cdot, \omega)$ is a probability measure on \mathbb{R} so that there exists $u^+(x, t, \omega) \in \mathbb{R}$ such that $f^+(x, t, \xi, \omega) = \mathbf{1}_{u^+(x, t, \omega) > \xi}$ for almost every (x, ξ, ω) . In particular, $u^+ = \int_{\mathbb{R}} (f^+ - \mathbf{1}_{\xi > 0}) d\xi$ for almost every (x, ω) . A similar result also holds for f^- .

It follows from the discussion after Definition 2.6 that f^+ being solution in the sense of Definition 2.6 implies that u^+ is a solution in the sense of Definition 2.3. Since $f = f^+$ a.e., this shows the reduction of reduction of generalized solutions to solutions. If now u_1 and u_2 are two solutions to (1.1), we deduce from (3.6) with $f_i = \mathbf{1}_{u_i > \xi}$ and from the identity

$$\int_{\mathbb{R}} \mathbf{1}_{u_1 > \xi} \overline{\mathbf{1}_{u_2 > \xi}} d\xi = (u_1 - u_2)^+$$

the contraction property

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{R}^d} \left[\tilde{K}(T - t, \cdot) * (u_1(t, \cdot) - u_2(t, \cdot))^+ \right] (x) dx \\ & \leq \mathbb{E} \int_{\mathbb{R}^d} \left[\tilde{K}(T, \cdot) * (u_1(0, \cdot) - u_2(0, \cdot))^+ \right] (x) dx, \end{aligned}$$

which implies the uniqueness of solutions. Actually, due to $K(T - x, t) > 0$ for any $t \in [0, T]$ and every $x \in \mathbb{R}^d$, we have if $u_{1,0} = u_{2,0}$, then $u_1 = u_2$.

In the multiplicative case (Φ depending on u), the reasoning is similar, except that there is an additional term in the bound on I_ψ . More precisely, by Hypothesis (1.4) we obtain in place of (3.4) the estimate

$$I_\psi \leq \frac{T\tilde{D}_1}{2} \epsilon^2 \delta^{-1} + \frac{D_1}{2} I_\psi^h,$$

where

$$I_\psi^h = \mathbb{E} \int_0^t \int_{\mathbb{R}^{2d}} \check{\rho}_\epsilon \int_{\mathbb{R}^2} \psi_\delta(\xi - \zeta) |\xi - \zeta| h(|\xi - \zeta|) d\nu_{x,\sigma}^1 \otimes \nu_{y,\sigma}^2(\xi, \zeta) dx dy d\sigma.$$

Choosing $\psi_\delta(\xi) = \delta^{-1} \psi_1(\delta^{-1} \xi)$ with ψ_1 compactly supported gives

$$I_\psi \leq \frac{T\tilde{D}_1}{2} \epsilon^2 \delta^{-1} + \frac{TD_1 C_\psi h(\delta)}{2}, \quad C_\psi := \sup_{\xi \in \mathbb{R}} \|\xi \psi_1(\xi)\|,$$

which implies that $\lim_{\epsilon \rightarrow 0, \delta = \epsilon^{4/3}} I_\psi = 0$. Similar to the additional case and the proof of Theorem 11 in [6], one can finish the proof of uniqueness of solution, which is the part of Theorem 3.1.

(Existence) We prove the existence by a vanishing viscosity method. Assume that $u_0 \in L^\infty \cap L^1 \cap BV(\mathbb{R}^d)$.

Consider the Cauchy problem:

$$\begin{cases} du^\epsilon(x, t) + [\operatorname{div}_x A(u^\epsilon) + (-\Delta_x)^{\frac{\alpha}{2}} u^\epsilon] dt \\ \quad - \epsilon \Delta u^\epsilon dt = \Phi^\epsilon(u^\epsilon) dW(t), \quad (x, t) \in (0, T) \times \mathbb{R}^d, \\ u^\epsilon(t = 0) = u_0, \quad x \in \mathbb{R}^d, \end{cases} \quad (3.7)$$

where Φ^ε is a suitable Lipschitz approximation of Φ satisfying (1.3) and (1.4) uniformly. We define g_k^ε and G_ε as in the case $\varepsilon = 0$.

Similar to the proof of Lemma 4.9 in [11], we can prove equation (3.7) has a unique solution $u^\varepsilon \in L^\infty([0, T], L^p(\mathbb{R}^d) \cap L^2([0, T], H^{\frac{\alpha}{2}}(\mathbb{R}^d)))$ provided that $u_0 \in L^p(\mathbb{R}^d)$, $p \geq 2$. Moreover, by using Itô formula, one can prove that u^ε satisfies the energy inequality

$$\begin{aligned}
& \mathbb{E} [\|u^\varepsilon(t)\|_2^2] + 2\varepsilon \mathbb{E} \int_0^t \int_{\mathbb{R}^d} |\nabla u^\varepsilon(s, x)|^2 dx ds \\
& + 2\mathbb{E} \int_0^t \int_{\mathbb{R}^d} u^\varepsilon(s, x) (-\Delta)^{\frac{\alpha}{2}} u^\varepsilon(s, x) dx ds \\
= & \mathbb{E} [\|u(0)\|_2^2] - 2\mathbb{E} \int_0^t \int_{\mathbb{R}^d} u^\varepsilon(s, x) \operatorname{div} A(u^\varepsilon(s, x)) dx ds \\
& + \mathbb{E} \int_0^t \int_{\mathbb{R}^d} G_\varepsilon^2(u^\varepsilon(s, x)) dx ds \\
\leq & \mathbb{E} [\|u(0)\|_2^2 + \|\hat{g}\|_{L^1}] + D_0 \int_0^t \mathbb{E} [\|u^\varepsilon(s)\|_2^2] ds,
\end{aligned}$$

which implies that by using Gronwall's Lemma

$$\begin{aligned}
& \mathbb{E} [\|u^\varepsilon(t)\|_2^2] + 2\varepsilon \mathbb{E} \int_0^t \|\nabla u^\varepsilon(s)\|_{L^2(\mathbb{R}^d)}^2 ds \\
& + 2\mathbb{E} \int_0^t \|u^\varepsilon(s)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}^2 ds \leq C_T \mathbb{E} [\|u(0)\|_2^2 + \|\hat{g}\|_{L^1}].
\end{aligned} \tag{3.8}$$

Also, for $p \geq 2$, by Itô formula applied to $|u^\varepsilon|^p$ and a martingale inequality

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|u^\varepsilon(t)\|_{L^p(\mathbb{R}^d)}^p \right) + \varepsilon \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |u^\varepsilon(x, t)|^{p-2} |\nabla u^\varepsilon(x, t)|^2 dx dt \leq C(p, u_0, T). \tag{3.9}$$

Similar to the discussion in Section 2 and the proof of Proposition 18 in [6], we can obtain the following result.

Proposition 3.1 (*Kinetic formulation*) *Let $u_0 \in L^\infty \cap L^1 \cap BV(\mathbb{R}^d)$ and let u^ε be the solution to (3.7). Then $f^\varepsilon := \mathbf{1}_{u^\varepsilon > \xi}$ satisfies: for all $\varphi \in C_c^2(\mathbb{R}^d \times [0, T] \times \mathbb{R})$,*

$$\begin{aligned}
& \int_0^T \langle f^\varepsilon(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f^\varepsilon(t), a(\xi) \cdot \nabla \varphi(t) + (-\Delta)^{\frac{\alpha}{2}} \varphi(t) - \varepsilon \Delta \varphi(t) \rangle dt \\
= & - \sum_{k \geq 1} \int_0^T \int_{\mathbb{R}^{d+1}} g_k^\varepsilon(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}^\varepsilon(\xi) dx d\beta_k(t) \\
& - \frac{1}{2} \int_0^T \int_{\mathbb{R}^{d+1}} \partial_\xi \varphi(x, t, \xi) G_\varepsilon^2(x, \xi) d\nu_{x,t}^\varepsilon(\xi) dx dt + m^\varepsilon(\partial_\xi \varphi),
\end{aligned} \tag{3.10}$$

a.s., where $f_0(\xi) = \mathbf{1}_{u_0 > \xi}$, $m^\varepsilon = m_1^\varepsilon + m_2^\varepsilon$, m_1^ε is defined as (2.4) and

$$\nu_{x,t}^\varepsilon = \delta_{u^\varepsilon(x,t)}, \quad m_2^\varepsilon = \varepsilon |\nabla u^\varepsilon(x, t)|^2 \delta_{u^\varepsilon(x,t)=\xi}.$$

Let η_ϵ satisfy the assumption in Definition 2.1 and $\eta_\epsilon(r) \rightarrow |r|$ as $\epsilon \rightarrow 0$. Itô formula gives

$$\begin{aligned} d\eta_\epsilon(u^\epsilon) &= -\eta'_\epsilon(u^\epsilon)[\operatorname{div}_x A(u^\epsilon) - (-\Delta_x)^{\frac{\alpha}{2}} u^\epsilon + \epsilon \Delta u^\epsilon] dt \\ &\quad + \eta'_\epsilon(u^\epsilon) \Phi^\epsilon(u^\epsilon) dW(t) + \frac{1}{2} \eta''_\epsilon(u^\epsilon) G_\epsilon^2 dt \end{aligned} \quad (3.11)$$

The convex of η implies that

$$\begin{aligned} \epsilon \eta'(u^\epsilon) \Delta u^\epsilon &= \epsilon \Delta \eta(u^\epsilon) - \epsilon \eta''(u^\epsilon) |\nabla u^\epsilon|^2 \leq \Delta \eta(u^\epsilon) \\ \eta'(u^\epsilon) (-\Delta_x)^{\frac{\alpha}{2}} u^\epsilon(x, t) &\geq c_0 \int_{\mathbb{R}^d} \frac{\eta(u^\epsilon(x, t)) - \eta(u^\epsilon(t, z + x))}{|z|^{d+\alpha}} dz = (-\Delta_x)^{\frac{\alpha}{2}} \eta(u^\epsilon(x, t)). \end{aligned}$$

Integrating (3.11) over \mathbb{R}^d , using the above two inequalities, taking expectation and letting $\epsilon \rightarrow 0$, we get

$$\begin{aligned} \mathbb{E} \int_{\mathbb{R}^d} |u^\epsilon(x, t)| dx &\leq \mathbb{E} \int_{\mathbb{R}^d} |u_0(x)| dx - \mathbb{E} \int_{\mathbb{R}^d} \operatorname{sgn}(u^\epsilon) [\operatorname{div}_x A(u^\epsilon)] dt \\ &\quad + \frac{1}{2} \lim_{\epsilon \rightarrow 0} \mathbb{E} \int_{\mathbb{R}^d} \eta''_\epsilon(u^\epsilon) G_\epsilon^2 dt \\ &\leq \|u_0\|_{L^1(\mathbb{R}^d)} + \frac{D_0}{2} \|\hat{g}\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

which implies that $u^\epsilon \in L^1(\mathbb{R}^d)$.

It follows from (3.8) that u^ϵ weakly converges in $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$.

Equation (3.11) is close to the kinetic equation (2.8) satisfied by the solution to (1.1). For $\epsilon \rightarrow 0$, we lose the precise structures of $m^\epsilon = \epsilon |\nabla u^\epsilon|^2 \delta_{u^\epsilon = \xi}$ and n^ϵ , and obtain a solution u to (1.1). More precisely, we will prove the following result.

Theorem 3.2 (*Convergence of the parabolic approximation*) *Let $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$. There exists a unique solution u to (1.1) with initial datum u_0 which is the strong limit of (u^ϵ) as $\epsilon \rightarrow 0$: for every $T > 0$, for every $1 \leq p < \infty$,*

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \|u^\epsilon - u\|_{L^p(\mathbb{R}^d \times (0, T))} = 0.$$

Moreover, (u^ϵ) converges weakly to u in $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$.

The proof of Theorem 3.2 is a straightforward consequence of both the result of reduction of generalized solution to solution (uniqueness of Theorem 3.1) and a priori estimates derived in the following.

Estimates of m_1^ϵ and m_2^ϵ : similar to that in [6], we analyze m_1^ϵ and m_2^ϵ . By (3.8), we obtain a uniform bound $\mathbb{E} m_2^\epsilon(\mathbb{R}^d \times [0, T] \times \mathbb{R}) \leq C$. Furthermore, the second term in the left hand-side of (3.9) is $\mathbb{E} \int_{\mathbb{R}^d \times [0, T] \times \mathbb{R}} |\xi|^{p-2} dm_2^\epsilon(x, t, \xi)$, so we have

$$\mathbb{E} \int_{\mathbb{R}^d \times [0, T] \times \mathbb{R}} |\xi|^p dm_2^\epsilon(x, t, \xi) \leq C_p.$$

We also have the improved estimate, for $p \geq 0$

$$\mathbb{E} \left| \int_{\mathbb{R}^d \times [0, T] \times \mathbb{R}} |\xi|^{2p} dm^\epsilon(x, t, \xi) \right|^2 \leq C_p, \quad (3.12)$$

where for $\psi \in C^2$ and $\psi'' \geq 0$,

$$\int_{\mathbb{R}^d \times [0, T] \times \mathbb{R}} \psi(\xi) dm_1^\varepsilon(x, t, \xi) = \int_{\mathbb{R}^d \times [0, T]} \left(\psi'(u^\varepsilon)(-\Delta)^{\frac{\alpha}{2}} u^\varepsilon - (-\Delta)^{\frac{\alpha}{2}} \psi(u^\varepsilon) \right) dx dt.$$

To prove (3.12), we apply Itô formula to $\psi(u^\varepsilon)$, $\psi(\xi) := |\xi|^{2p+2}$

$$\begin{aligned} & d\psi(u^\varepsilon) + \operatorname{div}(U)dt + \varepsilon \psi''(u^\varepsilon) |\nabla u^\varepsilon|^2 dt \\ & + [\psi'(u^\varepsilon)(-\Delta)^{\frac{\alpha}{2}} u^\varepsilon - (-\Delta)^{\frac{\alpha}{2}} \psi(u^\varepsilon)] dt \\ = & \psi'(u^\varepsilon) \Phi_\varepsilon(u^\varepsilon) dW + \frac{1}{2} \psi''(u^\varepsilon) G_\varepsilon^2 dt - (-\Delta)^{\frac{\alpha}{2}} \psi(u^\varepsilon) dt, \end{aligned}$$

where $U =: \int_0^{u^\varepsilon} a^\varepsilon(\xi) \psi'(\xi) d\xi - \varepsilon \nabla \psi(u^\varepsilon)$. Integrating over $\mathbb{R}^d \times [0, T]$ yields that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} [\varepsilon \psi''(u^\varepsilon) |\nabla u^\varepsilon|^2 + \psi'(u^\varepsilon)(-\Delta)^{\frac{\alpha}{2}} u^\varepsilon - (-\Delta)^{\frac{\alpha}{2}} \psi(u^\varepsilon)] dx dt \\ \leq & \int_{\mathbb{R}^d} \psi(u_0) dx + \sum_{k \geq 1} \int_0^T \int_{\mathbb{R}^d} \psi'(u^\varepsilon) g_{k, \varepsilon}(x, u^\varepsilon) dx d\beta_x(t) \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \psi''(u^\varepsilon) G_\varepsilon^2(x, u^\varepsilon) dx dt. \end{aligned}$$

Taking the square, then expectation, we deduce by Itô isometry

$$\begin{aligned} & \mathbb{E} \left| \int_0^T \int_{\mathbb{R}^d} [\varepsilon \psi''(u^\varepsilon) |\nabla u^\varepsilon|^2 + \psi'(u^\varepsilon)(-\Delta)^{\frac{\alpha}{2}} u^\varepsilon - (-\Delta)^{\frac{\alpha}{2}} \psi(u^\varepsilon)] dx dt \right|^2 \\ \leq & 3 \mathbb{E} \left| \int_{\mathbb{R}^d} \psi(u_0) dx \right|^2 + 3 \mathbb{E} \int_0^T \sum_{k \geq 1} \left| \int_{\mathbb{R}^d} \psi'(u^\varepsilon) g_{k, \varepsilon}(x, u^\varepsilon) dx \right|^2 dt \\ & + \frac{3}{2} \mathbb{E} \left| \int_0^T \int_{\mathbb{R}^d} \psi''(u^\varepsilon) G_\varepsilon^2(x, u^\varepsilon) dx dt \right|^2. \end{aligned}$$

By using Cauchy-Schwarz inequality and the condition (1.3), we obtain (3.12).

Estimate on ν^ε : This part is similar to that in [6]. Using the bound (3.9), we get

$$\mathbb{E} \sup_{t \in [0, T]} \int_{\mathbb{R}^d} \int_{\mathbb{R}} |\xi|^p d\nu_{x, t}^\varepsilon(\xi) dx \leq C_p \quad (3.13)$$

and, in particular,

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} |\xi|^p d\nu_{x, t}^\varepsilon(\xi) dx \leq C_p. \quad (3.14)$$

Consider a sequence $(\varepsilon_n) \downarrow 0$. First, by (3.14) and the proposition 2.1 (see [6, Theorem 5 and Corollary 6]), the convergence $\nu^{\varepsilon_n} \rightarrow \nu$ and $f^{\varepsilon_n} \rightharpoonup f$ in $L^\infty(\Omega \times \mathbb{R}^d \times (0, T) \times \mathbb{R})$ -weak-*. Besides, the bound (3.13) is stable: ν satisfies (2.7).

For $r \in \mathbb{N}^*$, let $K_r = \mathbb{R}^d \times [0, T] \times [-r, r]$ and let \mathcal{M}_r denote the space of bounded Borel measures over K_r (with norm given by the total variation of measures). It is the topological dual of $C(K_r)$, the set of continuous functions on K_r . Since \mathcal{M}_r is separable, the space $L^2(\Omega; \mathcal{M}_r)$ is the topological dual space of $L^2(\Omega, C(K_r))$, see Théorème 1.4.1 in [8]. The estimate (3.12)

with $p = 0$ gives a uniform bound on m^{ε_n} and n^{ε_n} in $L^2(\Omega; \mathcal{M}_r)$: there exists $m_r \in L^2(\Omega; \mathcal{M}_r)$ such that up to subsequence, $m^{\varepsilon_n} \rightharpoonup m$ in $L^2(\Omega; \mathcal{M}_r)$ -weak star. By a diagonal process, we obtain $m_r = m_{r+1}$ in $L^2(\Omega; \mathcal{M}_r)$ and the convergence in all the spaces $L^2(\Omega; \mathcal{M}_r)$ -weak star of a single subsequence still denoted (m^{ε_n}) . The condition at infinity (3.12) shows that m defines two elements of $L^2(\Omega; \mathcal{M})$, where \mathcal{M} denotes the space of bounded Borel measures over $\mathbb{R}^d \times (0, T) \times \mathbb{R}$. It follows that

$$\mathbb{E} \left| \int_{\mathbb{R}^d \times [0, T] \times \mathbb{R}} |\xi|^{2p} dm^\varepsilon(x, t, \xi) \right|^2 \leq C_p, \quad (3.15)$$

which is exactly as (45) in [6]. So following the idea of [6], we can prove the measure m satisfies 1, 2, and 3 in Definition 2.2, that is, m is a kinetic measure.

Proof of Theorem 3.2 By the proof of uniqueness, there corresponds a solution u to this $f : f = \mathbf{1}_{u > \xi}$. This proves the existence of a solution u to (1.1). Besides, owing to the particular structure of f^ε and f , we have

$$\|u^{\varepsilon_n}\|_{L^2(\mathbb{R}^d \times (0, T))}^2 - \|u\|_{L^2(\mathbb{R}^d \times (0, T))}^2 = \int_0^T \int_{\mathbb{R}^{d+1}} 2\xi(f^{\varepsilon_n} - f) d\xi dx dt$$

and (using the bound on u^ε in $L^3(\mathbb{R}^d)$)

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_{|\xi| > R} |2\xi(f^{\varepsilon_n} - f)| d\xi dx dt \leq \frac{C}{1 + R}.$$

It follows that u^{ε_n} converges in norm to u in the Hilbert space $L^2(\Omega \times \mathbb{R}^d \times (0, T))$. Using the weak convergence, we deduce the strong convergence. Since u is unique, the whole sequence actually converges. This gives the result of theorem for $p = 2$. The case of general p follows from the bound on u^ε in L^q for arbitrary q and Hölder inequality. Moreover, it follows from the uniform bound (3.8) that (u^ε) converges weakly to u in $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$. This completes the proof of Theorem 3.2. \square

The existence of solution in sense of Definition 2.2 are obtained by Theorem 3.2. This completes the proof of Theorem 3.1. \square

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References

- [1] N. Alibaud, *Entropy formulation for fractal conservation laws*, J. Evol. Equ. **7** (2007) 145-175.
- [2] C. Bauzet, G. Vallet and P. Wittbold, *The Cauchy problem for a conservation law with a multiplicative stochastic perturbation*, J. Hyperbolic Differential Equations **9** (2012) 661-709.
- [3] C. Bauzet, G. Vallet and P. Wittbold, *The Dirichlet problem for a conservation law with a multiplicative stochastic perturbation*, J. Funct. Anal. **266** (2014) 2503-2545.

- [4] I. Biswas and A. Majee, *Stochastic conservation laws: weak-in-time formulation and strong entropy condition*, J. Funct. Anal. **267** (2014) 2199-2252.
- [5] G. Q. Chen, Q. Ding and K. H. Karlsen, *On nonlinear stochastic balance laws*, Arch. Ration. Mech. Anal. **204** (2012) 707-743.
- [6] A. Debussche and J. Vovelle, *Scalar Conservation Laws with Stochastic Forcing*, J. Funct. Anal., **259** (2010) 1014-1042.
- [7] A. Debussche, M. Hofmanová and J. Vovelle, *Degenerate parabolic stochastic partial differential equations: quasilinear case*. Ann. Probab., **44** (2016) 1916-1955.
- [8] J. Droniou, *Intégration et Espaces de Sobolev à Valeurs Vectorielles*. (2001) <http://www-gm3.univ-mrs.fr/polys/>
- [9] J. Droniou and C. Imbert, *Fractal first-order partial differential equations*. Arch. Ration. Mech. An. 182(2) (2006) 299-331.
- [10] J. Duan, An Introduction to Stochastic Dynamics. Cambridge University Press, New York, 2015.
- [11] J. Feng and D. Nualart, *Stochastic Scalar Conservation Laws*, J. Funct. Anal., **255** (2008) 313-373.
- [12] I. Gyöngy and C. Rovira, *On L^p -solutions of semilinear stochastic partial differential equations*, Stochastic Process. Appl. **90** (2000) 83-108.
- [13] J. V. Kim, *On a stochastic scalar conservation law*, Indiana Univ. Math. J **52** (2003) 227-256.
- [14] P.L. Lions, B. Perthame and E. Tadmor, *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Am. Math. Soc. **7**(1) (1994) 169-191.
- [15] G. Lv, J. Duan and H. Gao, *Stochastic Nonlocal Conservation Laws I: Whole Space*, submitted.
- [16] G. Lv, J. Duan, H. Gao and J. Wu, *On a stochastic nonlocal conservation law in a bounded domain*, Bull. Sci. Math., **140** (2016) 718-746.
- [17] G. Vallet and P. Wittbold, *On a stochastic first-order hyperbolic equation in a bounded domain*, Infin. Dimens. Anal. Quantum Probab., **12** (2009) 1-39.
- [18] D.W. Stroock, *Diffusion processes associated with Lévy generators*, Z. Wahr. Verw. Geb. **32** (1975) 209-244.
- [19] M.F. Shlesinger, G.M. Zaslavsky and U. Frisch, Lévy Flights and Related Topics in Physics. Lecture Notes in Phys. 450, Springer-Verlag, Berlin, 1995.
- [20] J. Wei, J. Duan and G. Lv, *Kinetic solutions for nonlocal scalar conservation laws*, submitted.